

ON THE HILBERT GEOMETRY OF SIMPLICIAL TITS SETS

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ABSTRACT. The moduli space of convex projective structures on a hyperbolic simplicial Coxeter orbifold is either a point or can be identified with the open half real line. We study the Hilbert geometry of these convex projective structures. In particular, we prove that in the latter case, when one goes to 0 or infinity in the moduli space, the entropy of geodesic flow tends to 0.

1. STATEMENT OF THE RESULTS

Let \mathbb{P}^n be the real projective space of dimension n . A subset Ω of \mathbb{P}^n is called *convex* if either $\Omega = \mathbb{P}^n$ or Ω is contained in an affine chart and convex in that chart. If moreover the closure of Ω is contained in an affine chart, then Ω is called *properly convex*. Let $X = \tilde{X}/\Pi$ be an orbifold, where Π is a group which acts discontinuously on \tilde{X} . A *convex projective structure* on X is a convex projective realization of X , i.e., a faithful representation $\rho : \Pi \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$ together with a convex open subset $\Omega_\rho \subset \mathbb{P}^n$ such that there exists a ρ -equivariant homeomorphism from \tilde{X} to Ω_ρ . A convex projective structure ρ is called *strictly convex* if the boundary of Ω_ρ contains no segment. In particular, it is called *hyperbolic* if Ω_ρ is an ellipsoid.

We shall consider convex projective structures up to projective equivalence. The *moduli space of convex projective structures* on X , denoted by $\mathfrak{P}(X)$, is by definition the conjugacy classes of all representation ρ coming from convex projective structures. It is a subset in the space of conjugacy classes of representations $\mathfrak{R} = \mathrm{Hom}(\Pi, \mathrm{PGL}(n+1, \mathbb{R}))/\mathrm{PGL}(n+1, \mathbb{R})$. It is known that $\mathfrak{P}(X)$ is a connected component of \mathfrak{R} , see [3].

When X is a orientable closed surface of genus $g \geq 2$, Goldman [9] proved that $\mathfrak{P}(X)$ is homeomorphic to \mathbb{R}^{16g-16} .

In this article we study the case in which X is a *hyperbolic simplicial Coxeter orbifold*, namely, $X = \mathbb{H}^n/\Gamma$, where \mathbb{H}^n is the real hyperbolic n -space and $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is a *hyperbolic simplicial reflection group*, i.e., Γ is generated by the hyperbolic reflections with respect to the totally geodesic faces P_0, \dots, P_n of some compact simplex $P \subset \mathbb{H}^n$, such that P is a fundamental domain. In this case the dihedral angles of the faces determine a Coxeter graph J whose associated Coxeter group W_J is isomorphic to Γ . The following result should be well known and is stated in Goldman's survey [8] in the two-dimensional case.

Proposition 1. *Let $X_J = \mathbb{H}^n/\Gamma$, where $\Gamma \cong W_J$ is a hyperbolic simplicial reflection group, then*

$$\mathfrak{P}(X_J) \cong \begin{cases} \mathbb{R}_+ & \text{if the Coxeter graph } J \text{ has a circuit,} \\ \text{a point} & \text{otherwise.} \end{cases}$$

We will present a proof in Section 3 below for completeness.

We shall study how the convex set Ω_ρ deforms as ρ goes to 0 or $+\infty$ in the case $\mathfrak{P}(X_J) \cong \mathbb{R}_+$.

Proposition 2. *Let P be a simplex in \mathbb{P}^n . Let $X_J = \mathbb{H}^n/\Gamma$ be as in Proposition 1, such that $\mathfrak{P}(X_J) \cong \mathbb{R}_+$. Then there exists a one-parameter family of representations $\{\rho_t\}_{t \in \mathbb{R}_+}$ of W_J into $\mathrm{PGL}(n+1, \mathbb{R})$, such that*

- (1) *Each $\rho_t(W_J)$ is generated by projective reflections with respect to the faces of P .*
- (2) *$t \mapsto [\rho_t]$ is a homeomorphism from \mathbb{R}_+ to $\mathfrak{P}(X_J)$.*
- (3) *Let Ω_t be the convex open subset of \mathbb{P}^n preserved by $\rho_t(W_J)$. Then each Ω_t is properly convex, and Ω_t converges to P in the Hausdorff topology when t tends to 0 or $+\infty$. (See Figure 1)*

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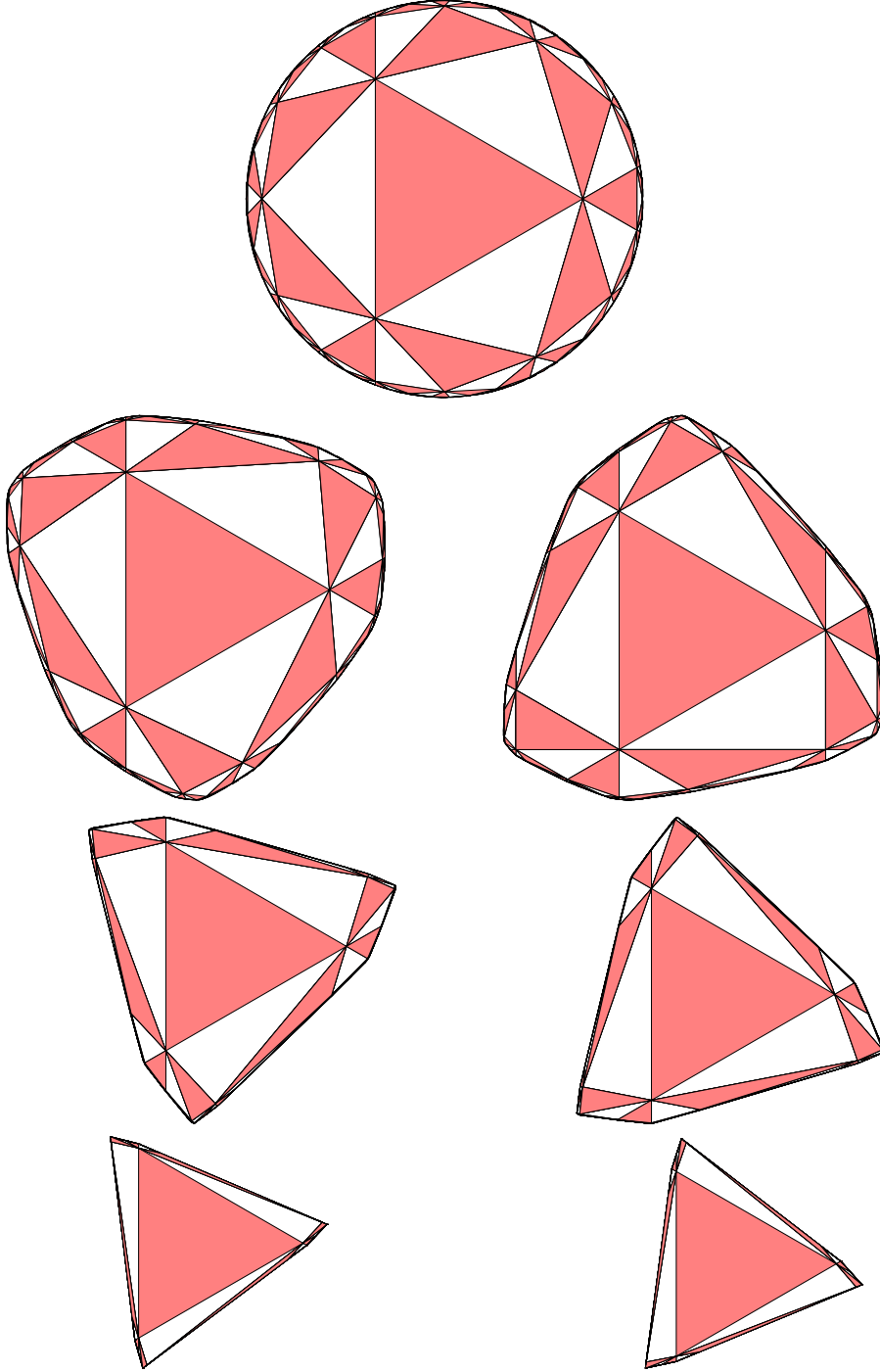


FIGURE 1. Deformation of Ω_t when t tends to 0 and $+\infty$. Here W_J is the $(4, 4, 4)$ -triangle group

Our main result concerns the metric geometry on the above family of convex projective orbifolds. Any properly convex set $\Omega \subset \mathbb{P}^n$ carries a canonical Finsler metric d_Ω , called the Hilbert metric. d_Ω is invariant under projective transformations, and thus provides canonical Finsler metrics on orbifolds covered by Ω . If Ω is an ellipsoid, then (Ω, d_Ω) is isometric to the real hyperbolic n -space \mathbb{H}^n .

The one-parameter family $\{\Omega_t\}$ in Proposition 2 give rise to a family $\{d_t\}$ of Hilbert metrics on the orbifold $X_J \cong \Omega_t/\rho_t(W_J)$. From Proposition 2 we can already deduce some easy geometric properties of this family. For example, the diameter and volume of X_J with respect to d_t tends to infinity as $t \rightarrow 0$ or ∞ . The purpose of this paper is to study some deeper geometric properties, namely, the entropy of geodesic flow.

For a compact non-positively curved Riemannian manifold X with universal covering \tilde{X} , A. Manning [14] proved that the topological entropy of geodesic flow on the unit tangent bundle of X equals the exponential growth of volume of balls in \tilde{X} , which also equals the “orbit growth” of any base point of \tilde{X} under the isometric $\pi_1(X)$ -action in the following sense:

Definition 1. Let Γ be a group, and (\tilde{X}, d) be a metric space. The orbit growth of a point $x_0 \in \tilde{X}$ with respect to an isometric discontinuous Γ -action on \tilde{X} is the number

$$\overline{\lim}_{R \rightarrow +\infty} \frac{1}{R} \log \# \Gamma x_0 \cap B(x_0, R)$$

where $B(x_0, R)$ denotes the ball of radius R centered at x_0 .

Note that if the stabilizer of any point of \tilde{X} is finite subgroup of Γ , then the orbit growth does not depend on the choice of x_0 .

The work of Manning can easily be generalized to convex projective manifolds, i.e., the topological entropy of geodesic flow of a compact convex projective manifold $X = \Omega/\Gamma$ equals the orbit growth of the Γ -action on the convex open set Ω with respect to the Hilbert metric d_Ω . M. Crampon [6] proved that for a fixed closed n -manifold X , this quantity is at most $n - 1$, while equality is achieved if and only if the projective structure is hyperbolic.

Theorem 1. Let X_J be a hyperbolic simplicial Coxeter orbifold with $\mathfrak{P}(X_J) \cong \mathbb{R}_+$, let $\rho_t : W_J \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$ and Ω_t be the family of representations and convex sets provided by Proposition 2. Denote the Hilbert metric on Ω_t by d_t . Then when $t \rightarrow 0$ or $+\infty$, the orbit growth of the W_J -action on (Ω_t, d_t) tends to 0.

Theorem 1 follows from a result on the large-scale geometry of (Ω_t, d_t) . First we recall some backgrounds. Two metric spaces (X_1, d_1) and (X_2, d_2) are called *quasi-isometric* if there are constants $A \leq 1, B \geq 1, M \geq 0$ and a map $f : X_1 \rightarrow X_2$ such that any point of X_2 has bounded distance with the image $f(X_1)$, and for all $x, x' \in X_1$ we have

$$Ad_2(f(x), f(x')) - M \leq d_1(x, x') \leq Bd_2(f(x), f(x')) + M$$

Definition 2. The infimum of $\log \frac{B}{A}$, where A and B satisfy the above condition, is called the constant of quasi-isometry between (X_1, d_1) and (X_2, d_2) .

Note that if $X_2 = X_1$ and d_2 is a dilation of d_1 , then the constant of quasi-isometry is zero. Generally, the constant of quasi-isometry measures how far (X_2, d_2) is from being a dilation of (X_1, d_1) “in large scale”.

The Švarc-Milnor lemma ([4], Proposition 8.19) says that each (Ω_t, d_t) is quasi-isometric to (W_J, d_J) , where d_J is the word-length-metric of W_J with respect to the standard generating set. In fact, for any $x_0 \in \Omega_t$, the orbital map $\gamma \mapsto \rho_t(\gamma)x_0$ is quasi-isometric. Our main result states that the constant of quasi-isometry between W_J and Ω_t is bounded independently of $t \in \mathbb{R}_+$.

Theorem 2. There is a positive constant β depending only on the Coxeter graph J , such that for any $t \in \mathbb{R}_+$, the constant of quasi-isometry between (W_J, d_J) and (Ω_t, d_t) is at most β .

In fact, we will see when t tends to 0 or $+\infty$, the coarse geometry of (Ω_t, d_t) resembles that of (W_J, d_J) dilated by a constant tending to $+\infty$.

The main ingredient of the proof of Theorem 1 and 2 is the following local result,

Proposition 3. There exists a constant C depending only on the Coxeter graph J , such that if A and B are two k -dimensional cells of P and $E = A \cap B$ is a $(k-1)$ -dimensional cell, then for any $x \in A, y \in B$ and any $t \in \mathbb{R}_+$, we have

$$Cd_t(x, y) \geq d_t(x, E) + d_t(y, E)$$

Theorem 2 and Proposition 3 are just manifestations of the reflectional symmetries of Ω_t . They are true not only for the Hilbert metrics d_t , but also for any W_J -invariant metric with the property that any two points in Ω_t can be joint by a unique geodesic.

As another consequence of Proposition 3, we construct interesting families of convex projective structures on closed surfaces.

Corollary 1. *On every oriented closed surface S of genus $g \geq 2$, there exists one-parameter family of convex projective structures $(\mathcal{P}_t)_{t \in \mathbb{R}_+}$, such that when $t \rightarrow 0$ or $+\infty$, the topological entropy of geodesic flow tends to 0, the systole tends to $+\infty$, and the constant of hyperbolicity tends to $+\infty$.*

Recall that for a closed metrized manifold X with universal covering \tilde{X} , the *systole* of X is defined to be the infimum of lengths of homotopically non-trivial closed curves on X , and the *constant of hyperbolicity* of X is defined to be the supremum of “sizes” of all geodesic triangles in \tilde{X} , where we define the size of a geodesic triangle Δ to be the minimal perimeter of all geodesic triangles inscribed to Δ . There are several equivalent definitions of size, see [10], Chapter 2, §3. X is called *δ -hyperbolic in the sense of Gromov* if it has constant of hyperbolicity $< +\infty$.

Here is the plan of the paper. In section 2, we give some backgrounds. In section 3, we prove Proposition 1 and 2. In section 4, admitting Proposition 3, we prove Theorem 1, 2 and Corollary 1. Finally we prove Proposition 3 in section 5.

2. PRELIMINARIES

In this section we recall some classical facts about reflection groups and Tits set. We refer to [1] chapter 1, [7] and [15] for details.

Let \mathbb{P}^n be the n -dimensional real projective space. By definition, a projective transformation $s \in \text{PGL}(n+1, \mathbb{R})$ is called a *reflection* if s is conjugate to $\pm \text{diag}(-1, 1, \dots, 1)$. A reflection s fixes a hyperplane $F \subset \mathbb{P}^n$ pointwisely, and has another fixed point f outside F . Reflections are in one-one correspondence with pairs (f, F) , where F is a hyperplane and f is a point outside F .

Given a n -dimensional simplex P in \mathbb{P}^n with faces P_0, \dots, P_n . We shall chose a reflection s_i with respect to each P_i and investigate the group $\Gamma \subset \text{PGL}(n+1, \mathbb{R})$ generated by $\{s_i\}_{i \in J}$. We call Γ a *simplicial reflection group*, and P the *fundamental simplex*. We will say the reflection group is *marked* if we want to keep track of the order of generators. Note that n -dimensional simplices are conjugate to each other by projective transformations. Since we do not want to distinct conjugate marked simplicial reflection groups, we can always assume that P is the simplex $\{[x_0 : \dots : x_n] \in \mathbb{P}^n | x_i \geq 0, \forall i\}$, while that faces $P_i = \{[x_0 : \dots : x_n] \in \mathbb{P}^n | x_i = 0, x_k \geq 0 \text{ for any } k \neq i\}$.

Once the fundamental simplex is given, the data we need for the determination of a marked reflection group are ordered $n+1$ points f_0, \dots, f_n satisfying $f_i \notin P_i$. Using the homogeneous coordinates, we suppose $f_i = [\alpha_{i0} : \dots : \alpha_{in}]$. By a normalization, we can assume $\alpha_{ii} = 1$ for any $0 \leq i \leq n$. We record these f_i 's by the matrix $\mathbf{A} = (\alpha_{ij})$. Namely, the diagonal entries of \mathbf{A} are 1, and the i -th row of \mathbf{A} gives the homogenous coordinates of f_i .

Two marked simplicial reflection groups, given by \mathbf{A} and \mathbf{A}' respectively, are conjugate if and only if there is a projective transformation which stabilizes P and takes f_i to f'_i . This is equivalent to the existence of positive numbers $\lambda_0, \dots, \lambda_n$ such that $\text{diag}(\lambda_0, \dots, \lambda_n) \mathbf{A} \text{diag}(\lambda_0^{-1}, \dots, \lambda_n^{-1}) = \mathbf{A}'$. We write $\mathbf{A} \sim \mathbf{A}'$ if \mathbf{A} and \mathbf{A}' satisfy the above condition. This defines an equivalent relation on \mathcal{M}^{n+1} , the set of $(n+1) \times (n+1)$ -matrices whose diagonals are 1. Then moduli space of marked simplicial reflection groups in \mathbb{P}^n is \mathcal{M}^{n+1} / \sim . However, a generic point in this space gives a non-discrete group.

Recall that a *Coxeter graph* J is a weighted graph determined by a finite set of nodes $\mathbf{v}_J = \{0, 1, \dots, n\}$ together with integers $m_{ij} \geq 2$ associated to each non-ordered pair of distinct nodes $i, j \in \mathbf{v}_J$. By convention, $m_{ij} = 2$ means there are no edges joining i and j . When $m_{ij} \geq 3$, means i and j are joint by an edge of weight m_{ij} .

The Coxeter graph $J = (\{0, 1, \dots, n\}, \{m_{ij}\})$ yields an abstract Coxeter group

$$W_J = \langle \tau_0, \dots, \tau_n | (\tau_i \tau_j)^{m_{ij}} = \tau_i^2 = \tau_j^2 = 1, \forall i \neq j \rangle$$

The *Cartan matrix* of J , denoted by \mathbf{C}_J , is defined to be the symmetric matrix whose diagonal entries are 1 and the (i, j) -entry is $-\cos(\pi/m_{ij})$ if $i \neq j$.

Remark. In the literature, when J comes from semi-simple Lie algebras, the usual definition of Cartan matrix is two times the above one.

Now we can state conditions under which $\mathbf{A} \in \mathcal{M}^{n+1}$ gives a discrete group. The following theorem is a special case of a more general result of Tits, see [1], Theorem 1.5.

Theorem (Tits). *Let $\mathbf{A} \in \mathcal{M}^{n+1}$ and f_i be the point in \mathbb{P}^n whose homogeneous coordinates is given by the i -th row of \mathbf{A} , where $i = 0, 1, \dots, n$. Let Γ be the subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ generated by s_0, \dots, s_n , where s_i is the reflection with respect to P_i and f_i .*

*Then the translates γP ($\gamma \in \Gamma$) are disjoint except at boundary if and only if there exists a Coxeter graph $J = (\{0, 1, \dots, n\}, \{m_{ij}\})$, such that \mathbf{A} satisfies the following condition $(*_J)$.*

$$(*_J) : \begin{array}{l} \text{For any distinct pair } i, j \in \{0, 1, \dots, n\}, \\ \text{we have } \alpha_{ij} \leq 0, \text{ and} \end{array} \quad \left\{ \begin{array}{ll} \alpha_{ij} = \alpha_{ji} = 0 & \text{if } m_{ij} = 2 \\ \alpha_{ij}\alpha_{ji} = \cos^2(\pi/m_{ij}) & \text{if } 3 \leq m_{ij} < \infty \\ \alpha_{ij}\alpha_{ji} \geq 1 & \text{if } m_{ij} \geq \infty \end{array} \right.$$

*When $(*_J)$ is satisfied, we have the following conclusions,*

- (1) $\rho : \tau_i \mapsto s_i$ ($0 \leq i \leq n$) is an isomorphism from W_J to Γ .
- (2) The set $\Omega = \cup_{\gamma \in \Gamma} \gamma P$, called the Tits set, is a convex subset of \mathbb{P}^n . Γ acts discontinuously on Ω .
- (3) Ω is open if and only if the stabilizer of each vertex of P is a finite group.

Observe that if $\mathbf{A} \sim \mathbf{B}$, then \mathbf{A} verifies $(*_J)$ if and only if \mathbf{B} does. Let $\mathcal{M}_J \subset \mathcal{M}^{n+1}$ be the set of matrices satisfying $(*_J)$ whose diagonals are 1. Then \mathcal{M}_J / \sim is the moduli space of marked reflection groups isomorphic to W_J .

We also observe that a subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ acts discontinuously on the whole \mathbb{P}^n if and only if it is a finite group. On the other hand, it is well known that W_J is a finite Coxeter group if and only if the Cartan matrix \mathbf{C}_J is positively definite (see [7]). So the Tits set Ω is a convex set contained in an affine chart if and only if J satisfies

- (i) \mathbf{C}_J is not positively definite.

Furthermore, the condition for openness in the theorem is equivalent to

- (ii) Every proper principle submatrix of \mathbf{C}_J is positively definite.

All the Coxeter graphs J satisfying (i) and (ii) are completely classified, see [7] or [15]. There are two cases:

Euclidean Case: In addition to (i) and (ii), we assume \mathbf{C}_J is degenerate. In this case \mathbf{C}_J has corank 1, and there is a faithful representation $\rho_0 : W_J \rightarrow \mathrm{Isom}(\mathbb{E}^n)$ which realize W_J as an Euclidean simplicial reflection group. In this case, the Tits set Ω is either a bigger simplex containing the simplex P , or an affine chart.

Hyperbolic Case: In addition to (i) and (ii), we assume \mathbf{C}_J is non-degenerate. Such a Coxeter graph is called a Lannér graph, as they are first classified by F. Lannér. In this case \mathbf{C}_J has signature $(1, n)$, and there is a faithful representation $\rho_0 : W_J \rightarrow \mathrm{Isom}(\mathbb{H}^n)$ which realize W_J as a hyperbolic simplicial reflection group. We reproduce in Figure 2 the table of all Lannér graphs (see [15], p.205). Note that they exist only for $n \leq 4$. In this case, since W_J is a hyperbolic group in the sense of Gromov, by the Theorem 1.1 of Benoist [2], the Tits set Ω is strictly convex, i.e., $\partial\Omega$ does not contain any straight line segment.

3. THE MODULI SPACE OF CONVEX PROJECTIVE STRUCTURES

In this section, we take a Lannér graph $J = (\{0, 1, \dots, n\}, \{m_{ij}\})$. Let $\rho_0 : W_J \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$ realizes W_J as a hyperbolic reflection group with fundamental simplex P . We will not distinguish W_J and its image $\rho_0(W_J)$. Our goal is to determine the space of convex projective structures on the orbifold $X_J = \mathbb{H}^n / W_J$.

Let P_0, \dots, P_n be the faces of P and L_i be the hyperplane of \mathbb{P}^n containing P_i . Consider a faithful representation $\rho : W_J \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$ which defines a convex projective structure. There is some convex open set Ω_ρ and a homeomorphism $\Phi : \mathbb{H}^n \rightarrow \Omega_\rho$ which is ρ -equivariant, i.e., $\Phi(\gamma.x) = \rho(\gamma).\Phi(x)$, for any $x \in \mathbb{H}^n$ and any $\gamma \in W_J$.

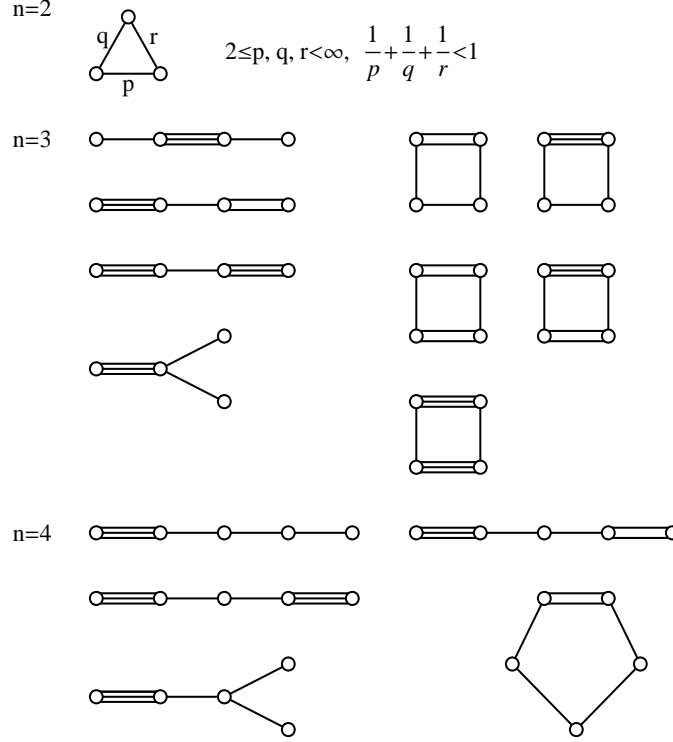


FIGURE 2. Lannér graphs. Here single edge, double edge and triple edge stand for weighted edge with weight 3, 4 and 5, respectively.

Since $\rho(\tau_i)$ has order 2, its fixed point set in \mathbb{P}^n is the disjoint union of a k -dimensional subspace and a $(n-k)$ -subspace. On the other hand, $\rho(\tau_i)$ fixes pointwisely $\Phi(L_i)$, a $(n-1)$ -dimensional submanifold of Ω_ρ , we conclude that $k=1$, $\Phi(L_i)$ is a $(n-1)$ -subspace and $\rho(\tau_i)$ is a reflection. Therefore, $\rho(W_J)$ is a projective reflection group with fundamental simplex $\Phi(P)$.

Following the discussion in the last section, we may suppose that the fundamental simplex is $P = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n | x_i \geq 0, \forall i\}$ and the reflection group $\rho(W_J)$ is given by some matrix $\mathbf{A} \in \mathcal{M}_J$, whose i -th row is the homogenous coordinates of a fixed point f_i of $\rho(\tau_i)$. Conversely, every $\mathbf{A} \in \mathcal{M}_J$ yields a representation $\rho : W_J \rightarrow \text{PGL}(n+1, \mathbb{R})$ preserving a Tits set Ω_ρ , and thus defines an element $[\rho] \in \mathfrak{P}(X_J)$. Moreover, given two such representations ρ_1 and ρ_2 which comes from $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{M}_J$ respectively, ρ_1 and ρ_2 are conjugate if and only if $\mathbf{A}_1 \sim \mathbf{A}_2$. Therefore, we have the identification

$$\mathfrak{P}(X_J) = \mathcal{M}_J / \sim$$

Now our task is to determined the latter quotient space. Given a $n \times n$ matrix $\mathbf{A} = (a_{ij})$ and an ordered set of indices $1 \leq i_1, \dots, i_k \leq n$ with $k \geq 1$, we call $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$ a *cyclic product of length k* .

Lemma 1. Let $\mathbf{A} = (a_{ij})$ be a $n \times n$ matrix satisfying the condition:

- (1) For any i , $a_{ii} \neq 0$. For any $i \neq j$, $a_{ij} = 0$ if and only if $a_{ji} = 0$

and the same hypothesis for \mathbf{B} . We write $\mathbf{A} \sim \mathbf{B}$ if \mathbf{A} and \mathbf{B} are conjugate via a diagonal matrix, i.e., there are $\lambda_1, \dots, \lambda_n \neq 0$, such that

$$\text{diag}(\lambda_1, \dots, \lambda_n) \mathbf{A} \text{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1}) = \mathbf{B}.$$

Then, $\mathbf{A} \sim \mathbf{B}$ if and only if their cyclic products with the same indices coincide, i.e., for any ordered subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, we have

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} = b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{k-1} i_k} b_{i_k i_1}.$$

Proof. We say a matrix \mathbf{A} is *reducible* if, after a reordering of basis if necessary, \mathbf{A} can be put into a block-diagonal form. Otherwise \mathbf{A} is said to be *irreducible*. The hypothesis on \mathbf{A} and \mathbf{B} implies $a_{ii} = b_{ii}$ and $a_{ij} = 0 \Leftrightarrow b_{ij} = 0$ for any $i \neq j$. Therefore, after a reordering of basis if necessary, we can assume that \mathbf{A} and \mathbf{B} are both block-diagonal with irreducible blocks, and the r -th block of \mathbf{A} has the same size with the r -th block of \mathbf{B} . Clearly, \mathbf{A} and \mathbf{B} are conjugate via a diagonal matrix if and only if their blocks are. Thus we can assume \mathbf{A} and \mathbf{B} are irreducible.

We are looking for $\lambda_1, \dots, \lambda_n$ which satisfy $\lambda_i^{-1} a_{ij} \lambda_j = b_{ij}$, or equivalently,

$$(2) \quad \frac{\lambda_i}{\lambda_j} = \frac{a_{ij}}{b_{ij}}, \text{ for all } i \neq j \text{ such that } b_{ij} \neq 0$$

First, we take $\lambda_1 = 1$. Irreducibility means that, for each $i \in \{1, 2, \dots, n\}$, there is sequence of distinct indices $1, i_1, i_2, \dots, i_k, i$, such that $a_{1i_1}, a_{i_1i_2}, \dots, a_{i_{k-1}i_k}, a_{i_ki}$ are all non-zero. We should set

$$(3) \quad \lambda_i = \frac{\lambda_1}{\lambda_{i_1}} \frac{\lambda_{i_1}}{\lambda_{i_2}} \dots \frac{\lambda_{i_{k-1}}}{\lambda_i} = \frac{a_{1i_1}}{b_{1i_1}} \frac{a_{i_1i_2}}{b_{i_1i_2}} \dots \frac{a_{i_{k-1}i_k}}{b_{i_{k-1}i_k}} \frac{a_{i_ki}}{b_{i_ki}}$$

this definition does not depend on the sequence of indices that we chose, since if we take another sequence $1, j_1, j_2, \dots, j_m, i$, then the definition becomes

$$(4) \quad \lambda_i = \frac{a_{1i_1}}{b_{1i_1}} \frac{a_{i_1i_2}}{b_{i_1i_2}} \dots \frac{a_{i_{k-1}i_k}}{b_{i_{k-1}i_k}} \frac{a_{i_ki}}{b_{i_ki}} = \frac{b_{1j_1}}{a_{1j_1}} \frac{b_{j_1j_2}}{a_{j_1j_2}} \dots \frac{b_{j_{m-1}j_m}}{a_{j_{m-1}j_m}} \frac{b_{j_mi}}{a_{j_mi}}$$

where we used the coincidence of cyclic products $a_{ij}a_{ji} = b_{ij}b_{ji}$. Now the hypothesis implies that the right hand sides of (3) and (4) are the same. In same way, we can verify that the hypothesis implies these λ_i 's satisfy (2). \square

Proof of Proposition 1. If there is no circuit in the Coxeter graph of J , then for any \mathbf{A} satisfying $(*_J)$, its cyclic products of length ≥ 3 are all 0, while cyclic products of length 1 are just diagonal entries, which equal 1, and cyclic products length 2 are determined by $(*_J)$. By Lemma 1, for any $\mathbf{A}, \mathbf{B} \in \mathcal{M}_J$, we have $\mathbf{A} \sim \mathbf{B}$.

If there is a circuit in the Coxeter graph, from Figure 2 we see that the whole graph is a circuit. Thus any \mathbf{A} satisfying $(*_J)$ has the following form (here we set $n = 4$, for example):

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_{01} & 0 & 0 & \alpha_{04} \\ \alpha_{10} & 1 & \alpha_{12} & 0 & 0 \\ 0 & \alpha_{21} & 1 & \alpha_{23} & 0 \\ 0 & 0 & \alpha_{32} & 1 & \alpha_{34} \\ \alpha_{40} & 0 & 0 & \alpha_{43} & 1 \end{pmatrix}$$

Again there are no choices for cyclic products of length 1 and 2. The only two non-zero cyclic products of length ≥ 3 are $\varphi(\mathbf{A}) = \alpha_{01}\alpha_{12}\dots\alpha_{n-1,n}\alpha_{n1}$ and $\tilde{\varphi}(\mathbf{A}) = \alpha_{10}\alpha_{21}\dots\alpha_{n,n-1}\alpha_{1n}$. But by the condition $(*_J)$ we have

$$\varphi(\mathbf{A})\tilde{\varphi}(\mathbf{A}) = \cos^2\left(\frac{\pi}{m_{01}}\right)\cos^2\left(\frac{\pi}{m_{12}}\right)\dots\cos^2\left(\frac{\pi}{m_{n1}}\right)$$

Therefore by Lemma 1, for any $\mathbf{A}, \mathbf{B} \in \mathcal{M}_J$, $\mathbf{A} \sim \mathbf{B}$ if and only if $\varphi(\mathbf{A}) = \varphi(\mathbf{B})$. This value is always positive if n is odd, and always negative if n is even. Thus the following map is a homeomorphism:

$$\begin{aligned} \mathfrak{P}(X_J) = \mathcal{M}_J / \sim &\rightarrow \mathbb{R}_+ \\ [\mathbf{A}] &\mapsto |\varphi(\mathbf{A})| \end{aligned}$$

\square

In order to study how the Tits set deforms when $[\mathbf{A}]$ goes to 0 or $+\infty$ in $\mathfrak{P}(X_J)$, we need the follow lemma, which bounds the Tits set by a simplex.

Lemma 2. *Let J be a Lannér graph and take $\mathbf{A} \in \mathcal{M}_J$. Let $f_i \in \mathbb{P}^n$ be a point whose homogeneous coordinates are given by the i -th row of \mathbf{A} . Consider the representation $\rho : W_J \rightarrow \text{PGL}(n+1, \mathbb{R})$ sending τ_i to the reflection s_i fixing P_i and f_i . Then, among the 2^n simplices of \mathbb{P}^n with vertex set $\{f_0, \dots, f_n\}$, there exists one whose interior contains the Tits set Ω .*

Proof. Let H_i be the hyperplane of \mathbb{P}^n passing through $f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n$. Assume by contradiction that Ω is not contained in any simplex with vertices f_0, \dots, f_n . Then Ω meets some H_i . Without loss of generality, we suppose $\Omega \cap H_0 \neq \emptyset$.

H_0 is stabilized by the finite Coxeter group $\Gamma_0 = \langle s_1, \dots, s_n \rangle$, because the reflection s_i stabilizes any hyperplane passing through f_i , and H_0 is spanned by f_1, \dots, f_n . Thus Γ_0 is a finite affine transformation group of the affine chart $\mathcal{A}_0 = \mathbb{P}^n \setminus L_0 \cong \mathbb{R}^n$. It follows that Γ_0 must have a fixed point in \mathcal{A}_0 , for instance, the barycenter of an orbit.

On the other hand, let L_i denotes the hyperplane containing P_i . Since the fixed point set of s_i is $L_i \cup \{f_i\}$, it is easy to see that the only common fixed point of s_1, \dots, s_n is $p_0 = [1 : 0 : \dots : 0] = L_1 \cap \dots \cap L_n$. So p_0 is exactly the fixed point of Γ_0 in \mathcal{A}_0 mentioned above, and we have proved $p_0 \in H_0$.

We may consider the affine chart \mathcal{A}_0 as a linear space with origin p_0 , and endow it with a Γ_0 -invariant Euclidean scalar product. Γ_0 is then a finite Euclidean Coxeter group generated by n Euclidean reflections with respect to the subspaces H_1, \dots, H_n . We shall remark that such a group can not preserve any convex cone except the whole \mathcal{A}_0 , since otherwise the barycenter p' of some non-zero orbit in the cone is a non-zero fixed point of the group, and it follows that each of H_1, \dots, H_n contains the line passing through p_0 and p' , contradicting the independence of the H_i 's.

Now take $C = \bigcup_{x \in \Omega \cap L_0} [p_0, x]$, the cone of $\Omega \cap H_0$ over p_0 . Here $[p_0, x]$ denotes the segment in Ω joining p_0 and x . C is a Γ_0 -invariant subset of Ω . $C \cap \mathcal{A}_0$ is a Γ_0 -invariant convex cone of \mathcal{A}_0 , clearly does not equals the whole \mathcal{A}_0 . Thus the above remark concludes our contradiction argument. \square

Proof of Proposition 2. We consider the case $n = 3$ to simplify the notation. Let us fix a Lannér graph J with 4 nodes which has a circuit. Any $\mathbf{A} \in \mathcal{M}_J$ has the form

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_{01} & 0 & \alpha_{03} \\ \alpha_{10} & 1 & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 1 & \alpha_{23} \\ \alpha_{30} & 0 & \alpha_{32} & 1 \end{pmatrix}, \text{ where } \alpha_{ij} < 0 \text{ and } \alpha_{ij}\alpha_{ji} = \cos^2(\pi/m_{ij}).$$

We define a one-parameter family of matrices $\{\mathbf{A}_t\}_{t \in \mathbb{R}_+} \subset \mathcal{M}_J$ as follows,

$$\mathbf{A}_t = \begin{pmatrix} 1 & -t \cos^2(\frac{\pi}{m_{01}}) & 0 & -t^{-1} \\ -t^{-1} & 1 & -t \cos^2(\frac{\pi}{m_{12}}) & 0 \\ 0 & -t^{-1} & 1 & -t \cos^2(\frac{\pi}{m_{23}}) \\ -t \cos^2(\frac{\pi}{m_{30}}) & 0 & -t^{-1} & 1 \end{pmatrix}$$

Since $|\varphi(\mathbf{A}_t)| = t^4$, by the proof of Proposition 1, every matrix \mathbf{A} in \mathcal{M}_J is \sim -equivalent to exactly one \mathbf{A}_t . $t \mapsto [\mathbf{A}_t]$ is a homeomorphism from \mathbb{R}_+ to \mathcal{M}_J / \sim .

For $i = 0, 1, 2, 3$, let $f_i(t)$ be the point in \mathbb{P}^n whose homogeneous coordinates are given by the i -th row of \mathbf{A}_t . Define $\rho_t : W_J \rightarrow \text{PGL}(n+1, \mathbb{R})$ to be the representation sending τ_i to the reflection fixing $f_i(t)$ and the face P_i of P . Then $t \mapsto [\rho_t]$ is a homeomorphism from \mathbb{R} to $\mathfrak{P}(X_J)$. Let Ω_t be the Tits set of $\rho_t(W_J)$.

$p_0 = [1 : 0 : 0 : 0], \dots, p_3 = [0 : 0 : 0 : 1]$ are the vertices of P . Each $f_i(t)$ converges to p_{i+1} when $t \rightarrow +\infty$, and to p_{i-1} when $t \rightarrow -\infty$. Here the indices are counted mod 4. Therefore the simplex containing Ω_t given by Lemma 2 converges to P in the Hausdorff topology when $t \rightarrow \pm\infty$. \square

4. METRIC GEOMETRY OF SIMPLICIAL TITS SETS

On any properly convex open set $\Omega \subset \mathbb{P}^n$, we define the *Hilbert metric* d_Ω as follows. Take any affine chart \mathcal{A} containing the closure of Ω . For $x, y \in \Omega$, let x_0, y_0 be the points on the boundary $\partial\Omega$ such that x_0, x, y, y_0 lie consecutively on the segment $[x_0, y_0]$, then we define

$$(5) \quad d_\Omega(x, y) = \frac{1}{2} \log[x_0, x, y, y_0], \text{ where } [x_0, x, y, y_0] = \frac{|x_0 - y||y_0 - x|}{|x_0 - x||y_0 - y|}$$

We refer to [2] or [5] for basic properties of the Hilbert metric. In this section, we study the geometry of the Hilbert metric d_t on Ω_t (see Proposition 2).

The goal of this section is to prove Theorem 1,2 and Corollary 1 admitting Proposition 3.

Ω_t has the structure of *Coxeter complex*, i.e., Ω_t is a simplicial complex whose k -cells are translates of the k -cells of P by the W_J -action. Denote the k -skeleton of Ω_t by $\Omega_t^{(k)}$. We introduce a family of auxiliary metrics $d_t^{(k)}$ as follows. Define $d_t^{(k)}$ to be the geodesic metric on $\Omega_t^{(k)}$ induced by d_t , i.e., $d_t^{(k)}(x, y)$ equals the minimal length of piecewise segments joining x, y and lying in $\Omega_t^{(k)}$. In particular, $d_t^{(n)}$ is just d_t , and $(\Omega_t^{(1)}, d_t^{(1)})$ is a metric graph.

Proposition 3 enables us to compare $d_t^{(1)}$ with the restriction of d_t to $\Omega_t^{(1)}$, as stated in the following lemma.

Lemma 3. *Suppose $2 \leq k \leq n$. There is a constant C , depending only on J , such that for any $t \in \mathbb{R}_+$ and any $x, y \in \Omega_t^{(k-1)}$, we have*

$$d_t^{(k)}(x, y) \leq d_t^{(k-1)}(x, y) \leq C d_t^{(k)}(x, y)$$

In particular, there is a constant C' depending only on J such that for any t and $x, y \in \Omega_t^{(1)}$,

$$(6) \quad d_t(x, y) \leq d_t^{(1)}(x, y) \leq C' d_t(x, y)$$

Proof. The first half of the inequality is evident from the definition.

We prove the second half. Let $c : [0, 1] \rightarrow \Omega_t^{(k)}$ be a piecewise segment joining $x, y \in \Omega_t^{(k-1)}$ such that the length of c equals $d_t^{(k)}(x, y)$.

Let $t_0 = 0, t_1, t_2, \dots, t_r = 1 \in [0, 1]$ be such that each $c([t_{i-1}, t_i])$ lies in a single k -cell, and the $c(t_i)$'s are in $\Omega_t^{(k-1)}$. Since c is length-minimizing, each $c([t_{i-1}, t_i])$ must be a segment, whose length equals the distance between then two end points. Thus if we could prove

$$d_t^{(k-1)}(c(t_{i-1}), c(t_i)) \leq C d_t^{(k)}(c(t_{i-1}), c(t_i))$$

then we would take the sum over $1 \leq i \leq r$ and use the triangle inequality to obtain

$$d_t^{(k-1)}(x, y) \leq C d_t^{(k)}(x, y)$$

Therefore, we can assume that both x and y lie on the boundary of a k -cell. Since each k -cell is isometric to some subcell of P , it is sufficient to prove that, for any k -dimensional subcell F of P , we have for all $t \in \mathbb{R}_+$ and all $x, y \in F$

$$d_t^{(k-1)}(x, y) \leq C d_t^{(k)}(x, y) = C d_t(x, y).$$

If x, y both lie on the same $(k-1)$ -dimensional subcell of F , then we have $d_t^{(k)}(x, y) = d_t^{(k-1)}(x, y)$ and there is nothing to prove. Thus, we can assume that $x \in A$ and $y \in B$, where A, B are $(k-1)$ -dimensional subcells of F , such that $E = A \cap B$ is a $(k-2)$ -dimensional subcell. Let $x_0, y_0 \in E$ be the nearest point to x, y in E , respectively. i.e., $d_t(x, E) = d_t(x, x_0)$ and $d_t(y, E) = d_t(y, y_0)$.

The three segments $[x, x_0]$, $[x_0, y_0]$ and $[y_0, y]$ lie in $\Omega_t^{(k-1)}$, and form a piecewise segment joining x, y . Thus we have

$$d_t^{(k-1)}(x, y) \leq d_t(x, x_0) + d_t(x_0, y_0) + d_t(y_0, y)$$

by the triangle inequality, we have

$$d_t(x_0, y_0) \leq d_t(x_0, x) + d_t(x, y) + d_t(y, y_0)$$

these two inequalities give

$$d_t^{(k-1)}(x, y) \leq 2d_t(x, x_0) + 2d_t(y, y_0) + d_t(x, y)$$

Now we apply Proposition 3, and conclude that

$$d_t^{(k-1)}(x, y) \leq (2C + 1)d_t(x, y)$$

this is the required inequality. \square

Proof of Theorem 2. It follows from Definition 2 that if the constants of quasi-isometry between X_1, X_2 and between X_2, X_3 is smaller than or equal to a and b respectively, then the constant of quasi-isometry between X_1 and X_3 is at most $a + b$. The inequality (6) in Lemma 3 implies that for any $t \in \mathbb{R}_+$, the constant of quasi-isometry between $(\Omega_t^{(1)}, d_t^{(1)})$ and (Ω_t, d_t) is at most C' . Thus, to prove Theorem 2, it is sufficient to show the constant of quasi-isometry between (W_J, d_J) and $(\Omega_t^{(1)}, d_t^{(1)})$ is also bounded independently of t . This can be done via another auxiliary metric. Recall that W_J -invariant geodesic metrics on the graph $\Omega_t^{(1)}$ are completely determined by lengths of the edges of P , and monotone with respect to each of these lengths. Let d be the metric on $\Omega_t^{(1)}$ defined by setting all edge lengths of P to be 1. Note that for $t \in \mathbb{R}_+$, all $(\Omega_t^{(1)}, d)$'s are in fact the same metric space. Let $l_{\max}(t)$ and $l_{\min}(t)$ denotes the length of the longest and the shortest edge of P under d_t respectively. By the monotonicity, we have for any t and $x, y \in \Omega_t^{(1)}$

$$(7) \quad l_{\min}(t)d(x, y) \leq d_t^{(1)}(x, y) \leq l_{\max}(t)d(x, y)$$

We claim that $l_{\max}(t)/l_{\min}(t)$ is bounded by a constant $b \geq 1$ which does not depend on t . Proposition 3 (only the special case $k = 1$ is needed here) implies that if two edges of P shares an end point then the proportion of their lengths under d_t is bounded from below and from above by positive constants. It follows that the proportion of lengths of any two edges, in particular $l_{\max}(t)/l_{\min}(t)$, is bounded independently of t .

It follows that the constant of quasi-isometry between the two metrics d and $d_t^{(1)}$ is at most $\log b$. On the other hand, since the natural W_J -action on $(\Omega_t^{(1)}, d)$ is isometric and cocompact, by the Švarc-Milnor lemma ([4], Proposition 8.19), (W_J, d_J) is quasi-isometric to $(\Omega_t^{(1)}, d)$ with some constant of quasi-isometry β' . Therefore, the constant of quasi-isometry between (W_J, d_J) and $(\Omega_t^{(1)}, d_t^{(1)})$ is at most $\beta' + \log b$. \square

Proof of Theorem 1. Using the notions introduced in the previous proof, we consider the orbit growths of the W_J -action on $\Omega_t^{(1)}$ with respect to the different metrics $d_t, d_t^{(1)}$ and d . Let us denote the three orbit growths by $h(d_t), h(d_t^{(1)})$ and $h(d)$ respectively. Our goal is to show $h(d_t) \rightarrow 0$ when $t \rightarrow 0$ or $+\infty$.

The Definition 1 of orbit growth and the comparisons of metrics (6) and (7) yields

$$\frac{1}{C'}h(d_t) \leq h(d_t^{(1)}) \leq h(d_t)$$

and

$$\frac{1}{l_{\max}(t)}h(d) \leq h(d_t^{(1)}) \leq \frac{1}{l_{\min}(t)}h(d)$$

Since the convex set Ω_t approaches P when $t \rightarrow 0$ or $+\infty$ by Proposition 2, by the definition of Hilbert metric (5), the length of any edge of P tends to $+\infty$, so $l_{\min}(t)$ tends to $+\infty$. Thus the two inequalities above implies $h(d_t) \rightarrow 0$. \square

Proof of Corollary 1. We take positive integers p, q, r with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, and assume for the moment that there is a subgroup of finite index Π in the (p, q, r) -triangle group $\Delta = \Delta_{p,q,r}$ such that Π acts freely on the hyperbolic plane \mathbb{H}^2 and the quotient \mathbb{H}^2/Π is a closed surface S . If each of p, q, r is larger than or equal to 3, then Proposition 1 and 2 provide an one-parameter-family of representations $\rho_t : W_J \rightarrow \text{PGL}(n+1, \mathbb{R})$ and $\rho_t(W_J)$ -invariant convex open set Ω_t . These give a family of convex projective structures on S , which we denote by \mathcal{P}_t . We shall show \mathcal{P}_t fulfils the needed properties.

As mentioned in the first section, the topological entropy of geodesic flow of (S, \mathcal{P}_t) equals the orbit growth of the Π -action on Ω_t . Since Π has finite index in Δ , this is the same as the orbit growth of the Δ -action, which tends to 0 by Theorem 1.

Proposition 3 implies that for any t , every triangle inscribed to the fundamental triangle P has perimeter greater than $\frac{1}{C}$ times the perimeter of P . When $t \rightarrow 0$ or $+\infty$, under the metric d_t , the length of each edge of P tends to $+\infty$, so the perimeters of all inscribed triangles tends to $+\infty$ uniformly. By the definition of constant of hyperbolicity, it follows that the constant of hyperbolicity of Ω_t tends to $+\infty$.

Now consider the systole of (S, \mathcal{P}_t) . We take a homotopically non-trivial closed curve c which is the shortest under d_t . The free homotopy class of c corresponds to a conjugacy class in Π , which may be represented by a word w of the generators τ_i, τ_2, τ_3 of Δ . The action of $\rho_t(w)$ on Ω_t does not have fixed points, but stabilizes a line $\tilde{c} \subset \Omega_t$ which covers c . Actually, $\rho_t(w)$ may be thought of as a translation of Ω_t along the axe \tilde{c} . Note that any word conjugate to w in Δ must contains all of the three letters τ_1, τ_2 and τ_3 , otherwise the $\rho_t(w)$ has a fixed point in Ω_t .

Hence the image of c by the orbifold covering map $S \cong \mathbb{H}^2/\Pi \rightarrow \mathbb{H}^2/\Delta \cong P$ is a closed billiard trajectory in P which hits all of the three sides. The lengths under d_t of such trajectories are bounded from below by the minimal perimeter of inscribed triangles of P . We have already seen the latter goes to $+\infty$. Therefore, the systole of (S, \mathcal{P}_t) goes to $+\infty$.

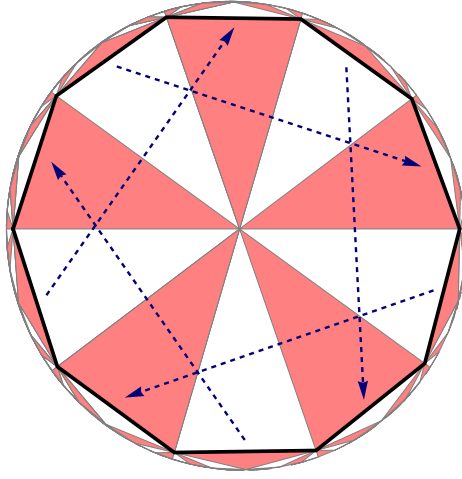


FIGURE 3. A subgroup of $\Delta_{5,5,5}$

Finally, our construction of projective structures on surface depends on a realization of the close surface S as quotient of \mathbb{H}^2 by a subgroup of some $\Delta_{p,q,r}$, with $p, q, r \geq 3$. Using straightforward constructions we may see that every surface of genus ≥ 2 can be realized in this way, so Corollary 1 is proved. Here is a construction:

Let $\Delta = \Delta_{5,5,5}$. Take five elements in Δ as shown in Figure 3, each of them sending the boldfaced 10-gon to a adjacent 10-gon. One may check that the group Π generated by these five elements has the 10-gon as a fundamental domain. The quotient \mathbb{H}^2/Π is a surface obtained by pairwise gluing edges of the 10-gon. A calculation of Euler characteristic shows \mathbb{H}^2/Π have genus 2. Since closed surfaces of higher genus covers the surface of genus 2, by taking subgroups of Π , we conclude that all surfaces of genus ≥ 2 is the quotient of \mathbb{H}^2 by some subgroup of Δ . \square

5. PROOF OF PROPOSITION 3

To begin with, we need the following fact concerning the simplicial complex structure and projective structure on Ω_t . Looking at Figure 1, we may observe that the 1-skeleton of Ω_t consists of straight lines. There is a same phenomenon in higher dimension, i.e., $\Omega_t^{(k)}$ is a union of k -dimensional subspaces. Note that by “subspace” of Ω_t , we mean the intersection of a subspace of \mathbb{P}^n with Ω_t . An equivalent statement of the above fact is that the k -dimensional subspace L containing a k -cell F must be an union of k -cells. This can be proved using the fact that the tangent space of a vertex in Ω_t has the structure of a finite Coxeter complex, and it is well known that the above statement holds for finite Coxeter complex (See for instance [12]). We omit the details.

First we present a proof of Proposition 3 for the 2-dimensional case, since the main idea is transparent in this case, while in higher dimensions we have to deal with some extra difficulties.

Proof of Proposition 3 for $n = 2$. We may assume

$$W_J = \langle \tau_1, \tau_2, \tau_3 | (\tau_1 \tau_2)^p = (\tau_2 \tau_3)^q = (\tau_3 \tau_1)^r = \tau_1^2 = \tau_2^2 = \tau_3^2 = 1 \rangle$$

Suppose x and y lie on the sides A and B of a triangle P in \mathbb{P}^2 , respectively. Denote the common vertex of A and B by E . We need to prove there is a constant C depending only on p, q, r , such that $Cd_t(x, y) \geq d_t(x, E) + d_t(y, E)$ for any t .

Let us fix a t and denote $s_1 := \rho_t(\tau_1)$, $s_2 := \rho_t(\tau_2)$, which are reflections with respect to A and B , respectively. $s_1 s_2$ is a rotation of order p .

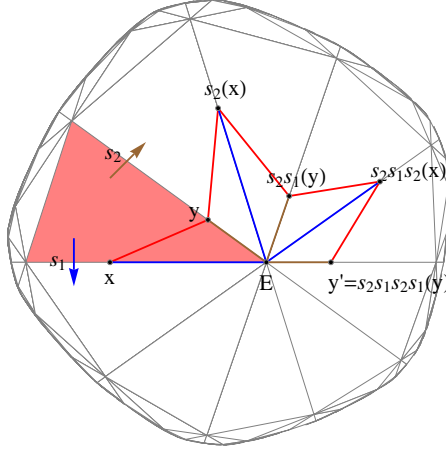


FIGURE 4. p odd
(here $p = 5$)

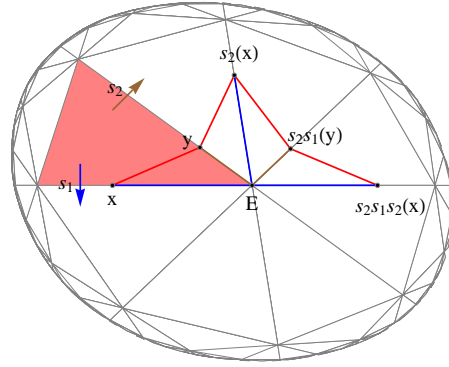


FIGURE 5. p even
(here $p = 4$)

When p is odd, $y' = \underbrace{s_2 s_1 s_2 \cdots s_1}_{p-1}(y)$ lies on the opposite half ray of the geodesic ray $\vec{E}x$ (see Figure 4). On the other hand, the successive images of $[x, y]$ by the sequence of transformations

$$s_2, s_2 s_1, s_2 s_1 s_2, \dots, \underbrace{s_2 s_1 s_2 \cdots s_1}_{p-1}$$

is a piecewise segment joining x and y' , which consists of p pieces, each piece having the same length $d_t(x, y)$. Thus we have

$$pd_t(x, y) \geq d_t(x, y') \geq d_t(x, E)$$

When p is even, we obtain $pd_t(x, y) \geq d_t(x, E)$ in the same way (see Figure 5).

As for y , we have the same inequality

$$pd_t(x, y) \geq d_t(y, E)$$

Thus we may conclude that

$$2pd_t(x, y) \geq d_t(x, E) + d_t(y, E)$$

Proposition 3 is proved for $n = 2$. \square

We introduce the following terminology. Let E be a $(k-1)$ -cell of Ω_t . We say two k -cells are *E-colinear*, if they lie on the same k -dimensional subspace and their intersection is E . As we have explained in the beginning of this section, the k -dimensional subspace of Ω_t containing a k -cell A is an union of k -cells. Thus for any $(k-1)$ -dimensional subcell E of A , there is a unique k -cell which is *E-colinear* to A .

The crucial point of the above proof is the following: let V be the k -cell *E-colinear* to A . Then we can connect $x \in A$ and some point of V by a curve which is piecewise isometric to $[x, y]$, and the number of pieces is determined combinatorially. We then have proved the needed inequality using the fact that the distance from x to any point of V is greater than the distance from x to E .

In higher dimensions, the situation is more delicate: the cell V which is E -colinear to A may not be a translate of A or B . In this case, we can not construct a curve piecewise isometric to $[x, y]$ going from x to V . Instead of this, we take a cell $A' = \rho_t(\gamma)A$, the translate of A by the action of some $\gamma \in W_J$, such that A' and V are contained in the same top-dimensional cell. Now we can go from x to A' along a curve piecewise isometric to $[x, y]$. To prove Proposition 3, we then need to show that the distance from x to A' is greater than the distance from x to E . In order to do this, we shall develop some lemmas concerning distance comparisons in Hilbert geometry.

Using the definition of Hilbert metric (5), it can be shown that if $\Omega \subset \mathbb{P}^n$ is a properly convex open set which is strictly convex (see the end of Section 2), then the Hilbert metric d_Ω has the following property. Let L be a subspace of arbitrary dimension of Ω and x be a point of Ω outside L . Then among all points of L , there is an unique point $x_0 \in L$ whose distance to x is minimal. We call x_0 the *projection* of x on L , and denote it by $x_0 = \text{Pr}(x, L)$.

Let $L \subset \Omega$ be a hyperplane, i.e., subspace of codimension one. We say that Ω have reflectional symmetry s with respect to L if $s \in \text{PGL}(n+1, \mathbb{R})$ is a reflection preserving Ω and fixing each point of L . In this case, the triangle inequality and the fact that geodesics are straight lines yields the following simple characterization of projection:

$$(8) \quad \text{Pr}(x, L) = [x, s(x)] \cap L$$

Lemma 4. *Let $\Omega \subset \mathbb{P}^n$ be a properly strictly convex open set with reflectional symmetry s with respect to a hyperplane L . Then for any $x, y \in \Omega$, we have*

$$d_\Omega(\text{Pr}(x, L), \text{Pr}(y, L)) \leq d_\Omega(x, y)$$

In particular, if $x \in L$, then for any $y \in \Omega$ we have

$$d_\Omega(x, \text{Pr}(y, L)) \leq d_\Omega(x, y)$$

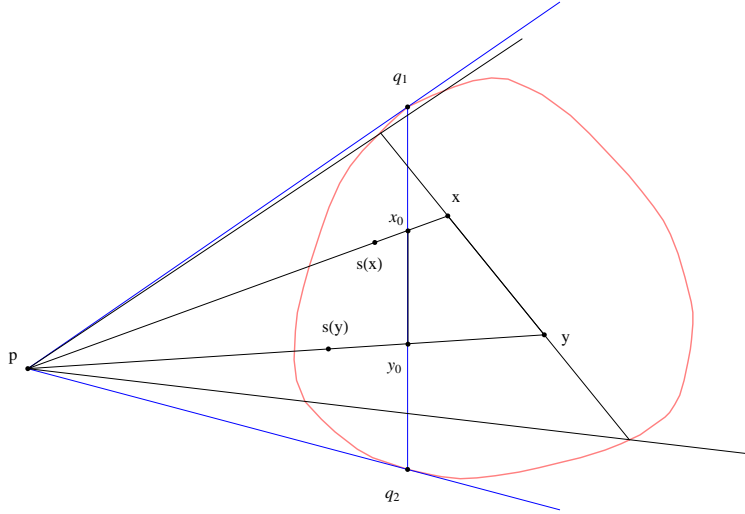


FIGURE 6. $d_\Omega(x_0, y_0) \leq d_\Omega(x, y)$

Proof. Denote $x_0 = \text{Pr}(x, L)$ and $y_0 = \text{Pr}(y, L)$. The reflection s has another fix point $p \in \mathbb{P}^n$ outside L . The reflection image $s(x)$ of x lies on the line \overline{xp} , and we have $x_0 = \overline{xp} \cap L$. The points y and y_0 has the same properties. Therefore, all the four points x, y, x_0, y_0 lie on the plane \overline{pxy} . This plane is invariant by s . So we may consider $\Omega_0 = \overline{pxy} \cap \Omega$ instead of Ω , and $L_0 = \overline{pxy} \cap L$ instead of L . Thus we have reduced to the two-dimensional case. See Figure 6

Suppose L_0 intersects $\partial\Omega_0$ at two points q_1, q_2 . Since Ω_0 has reflectional symmetry with respect to L_0 , the lines $\overline{pq_1}$ and $\overline{pq_2}$ are tangent to Ω . Now the inequality $d_\Omega(x_0, y_0) \leq d_\Omega(x, y)$ follows from

the definition (5) of Hilbert metric and the following well known fact from projective geometry (see for instance [5]): given four lines l_i ($1 \leq i \leq 4$) meeting at a point p , then for any line l intersecting l_1, l_2, l_3, l_4 consecutively at p_1, p_2, p_3, p_4 , the number $[p_1, p_2, p_3, p_4]$ is a constant not depending on the choice of l .

□

Remark. Lemma 4 is not true without the hypothesis of symmetry.

Lemma 5. *Let $\Omega \subset \mathbb{P}^n$ be a properly strictly convex open set with reflectional symmetries s_1, \dots, s_m with respect to hyperplanes L_1, L_2, \dots, L_m , such that s_1, \dots, s_m generates a finite group Γ . Assume $W = L_1 \cap \dots \cap L_m$ has dimension ≥ 1 and $W \cap \Omega \neq \emptyset$. Let D be a Γ -invariant convex subset of Ω .*

Then for any point x of W outside D and any $x' \in D$, there is some point $x_0 \in D \cap W$ such that

$$d_\Omega(x, x_0) \leq d_\Omega(x, x')$$

Proof. Fix a point $x \in D$. We chose an affine chart $\mathcal{A} \subset \mathbb{P}^n$, an origin point x_0 of \mathcal{A} in order to endow \mathcal{A} with a linear space structure, and then an Euclidean scalar product on \mathcal{A} . We could make these choices so as to fulfil the following conditions:

- (i) \mathcal{A} contains the closure of Ω .
- (ii) L_1, \dots, L_m are linear subspaces of \mathcal{A} . In particular, the origin x_0 of \mathcal{A} lies in W .
- (iii) Γ preserves the Euclidean scalar product.
- (iv) $x' \in W^\perp$, where W^\perp is the orthogonal complement of W .

Our aim is to show that $x_0 \in D$ and x_0 satisfies the required inequality. We shall consider intensively the linear space W^\perp . Let us denote the origin x_0 simply as 0. Each of $L'_i = L_i \cap W^\perp$ is a subspace of W^\perp of codimension 1, and the intersection $L'_1 \cap \dots \cap L'_m = \{0\}$. Since $D \cap W^\perp$ is Γ -invariant and convex, the barycenter of the Γ -orbit of x lies in $D \cap W^\perp$ and is fixed by Γ . But $L'_1 \cap \dots \cap L'_m = \{0\}$ implies the only fixed point of Γ in W^\perp is 0, thus $x_0 \in \Omega$.

For $1 \leq i \leq m$, we define

$$C_i = \{y \in W^\perp \mid \angle(y, L'_i) \geq \theta, \text{ or } y = 0\}$$

where $\angle(y, L'_i)$ is the usual Euclidean angle. We may take θ small enough so that $C_1 \cup \dots \cup C_m = W^\perp$. Any $y \in C_i$ verifies

$$|\text{Pr}_{W^\perp}(y, L'_i)| \leq |y| \cos \theta$$

where $\text{Pr}_{W^\perp}(y, L'_i)$ is the usual Euclidean projection of y on L'_i . Using the characterization of projection onto reflectional hyperplanes (8), we see that $\text{Pr}_{W^\perp}(y, L'_i)$ coincides with the projection $\text{Pr}(y, L_i)$ in the sense of Hilbert geometry described earlier.

We construct a sequence of points $x' = y_0, y_1, y_2, \dots \in D \cap W^\perp$ converging to 0 by recurrence as follows. Since $C_1 \cup \dots \cup C_m = W^\perp$, there is some C_{i_k} containing y_k . Then we set $y_{k+1} = \text{Pr}_{W^\perp}(y_k, L'_{i_k})$. The above inequality yields

$$|y_k| \leq |y_{k-1}| \cos \theta \leq \dots \leq |y_0| \cos^k \theta$$

Hence y_k converges to x_0 as $k \rightarrow \infty$.

As mentioned above, $y_{k+1} = \text{Pr}(y_k, L_{i_k})$. Lemma 4 implies

$$d_\Omega(x, y_k) \leq d_\Omega(x, y_{k-1}) \leq \dots \leq d_\Omega(x, y_0) = d_\Omega(x, x')$$

Therefore, by the continuity of d_Ω , we conclude that

$$d_\Omega(x, x_0) = \lim_{k \rightarrow \infty} d_\Omega(x, y_k) \leq d_\Omega(x, x')$$

□

Let us return to the particular convex set Ω_t . For any cell V of Ω_t , the union of all n -cells containing V is called the star-like neighborhood of V , and denoted by $\text{St}(V)$. We need the following

Lemma 6. *$\text{St}(V)$ is a convex subset of Ω_t .*

Proof. Let F be any $(n-1)$ -cell lying on the boundary of $\text{St}(V)$, and let L be the hyperplane containing F . L does not contain V , so V is contained in exactly one of the two closed “half spaces” of Ω_t bounded by L . Since $\text{St}(V)$ is an union of n -cells containing V , using the fact that L is an union of $(n-1)$ -cells, we can conclude the whole $\text{St}(V)$ lie in the same half space as V .

Thus, $\text{St}(V)$ is an intersection of closed half spaces, therefore convex. \square

Proof of Proposition 3 in general case. Fix a Lannér graph J . By definition of Lannér graphs, a subgroup of $W_J = \langle \tau_0, \dots, \tau_n | (\tau_i \tau_j)^{m_{ij}} = \tau_i^2 = 1 \rangle$ generated by a proper subset of $\{\tau_0, \dots, \tau_n\}$ is a finite Coxeter group. Let C be the maximum of word-length-diameters of all such subgroups. We will show $Cd_t(x, y) \geq d_t(x, E)$. Then by exchanging the roles of x and y , we have $Cd_t(x, y) \geq d_t(y, E)$, and these two inequalities give the required one.

Let V be the k -cell which is E -colinear to A . There is $\gamma \in \text{Stab}_{\rho(W_J)}(E)$, such that $P' = \rho_t(\gamma)P$ is a top-dimensional cell containing V . We denote $A' := \rho_t(\gamma)A \subset P'$, $x' := \rho_t(\gamma)x \in A'$.

First we claim that there is a curve joining x and x' which is piecewise isometric to $[x, y]$, with number of pieces at most C .

Denote $s_i = \rho_t(\tau_i)$, a reflection with respect to the face P_i of P . Let $J_E \subset \{0, 1, \dots, n\}$ be the set of indices of those P_i such that $E \subset P_i$. Then $\#J_E = n - k + 1$, and $\text{Stab}_{\rho(W_J)}(E)$ is generated by $\{s_i\}_{i \in J_E}$.

We can write $\rho_t(\gamma) = s_{i_1} s_{i_2} \dots s_{i_m}$, with $i_1, \dots, i_m \in J_E$ and $m \leq C$. Consider the sequence of segments

$$s_{i_1}([x, y]), s_{i_1} s_{i_2}([x, y]), \dots, s_{i_1} s_{i_2} \dots s_{i_m}([x, y]).$$

The k -cell A contains the $(k-1)$ -cell E , so there is only one vertex a of A which lies outside E . Similarly B has only one vertex b outside E . Each face P_i of P must contain at least one of the two points a and b . Hence each face containing E also contains A or B . It follows that if $i \in J_E$ then s_i fixes x or y . Therefore, each segment in the above sequence shares at least one end point with the next one. So the union of these segments is connected, and we can extract a subset of these segments to form a curve joining x and $x' = \rho_t(\gamma)x$ which is piecewise isometric to $[x, y]$. The number of pieces is at most m , hence bounded by C .

Thus, we conclude

$$Cd_t(x, y) \geq d_t(x, x')$$

Next, we need to prove

$$(9) \quad d_t(x, x') \geq d_t(x, E)$$

We apply Lemma 5. Let $\text{St}(V)$ be the convex compact set D in the hypothesis of Lemma 5, which contains x' . Let $J_A \subset \{0, 1, \dots, n\}$ be the set of indices of those faces P_i such that $A \subset P_i$, and let L_i be the hyperplane on which P_i lies. Then $W = \cap_{i \in J_A} L_i$ is the k -dimensional subspace containing A and V . For each $i \in J_A$, since L_i contains V , the reflection s_i preserves $\text{St}(V)$. Thus the hypothesis of Lemma 5 are verified, and we conclude that there is $x_0 \in V$ such that

$$d_t(x, x') \geq d_t(x, x_0)$$

$A \cup V$ is the intersection of $\text{St}(E)$ and a k -dimensional subspace, thus must be convex. So $[x, x_0]$ intersects E at some point x_1 . Clearly we have

$$d_t(x, x_0) \geq d_t(x, x_1) \geq d_t(x, E)$$

Hence we have obtained (9), and the proof is complete. \square

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